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Cosmic acceleration from asymmetric branes

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Abstract

We consider a single 3-brane sitting in between two different five dimensional spacetimes. On each side of the brane, the bulk is a solution to Gauss-Bonnet gravity, although the bare cosmological constant, fundamental Planck scale, and Gauss-Bonnet coupling can differ. This asymmetry leads to weighted junction conditions across the brane and interesting brane cosmology. We focus on two special cases: a generalized Randall-Sundrum model without any Gauss-Bonnet terms, and a stringy model, without any bare cosmological constants, and positive Gauss-Bonnet coupling. Even though we assume there is no vacuum energy on the brane, we find late time de Sitter cosmologies can occur. Remarkably, in certain parameter regions, this acceleration is preceded by a period of matter/radiation domination, with $H^2 \propto \rho$, all the way back to nucleosynthesis.

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1 Introduction

Recent observations suggest that our universe is accelerating [1, 2]. In the standard cosmology, such an acceleration cannot be driven by ordinary matter and radiation. The simplest explanation is to imagine that there is some form of positive vacuum energy/cosmological constant. Although some progress has been made recently [3, 4, 5], it is notoriously difficult to produce a positive cosmological constant from compactifications of string/M-theory. Given that string theory is currently our best candidate for a quantum theory of gravity, this is a big worry. It is natural, therefore, to seek other explanations.

The standard technique is to modify Einstein gravity is some particular way. In quintessence theories we add a scalar field to obtain the desired acceleration [6]. However, none of these theories appear to be related to a more fundamental theory of quantum gravity. Another solution is to consider theories that exhibit infra-red modifications of gravity [7, 8, 9, 10, 11, 12]. Generically, these also lead to cosmic acceleration at late times [13, 14, 15, 16, 17, 18]. Perhaps the most celebrated of these theories are the DGP model [7], multigravity [9], and more recently the idea of a ghost condensate [11, 12]. The DGP model is a braneworld model where there is a large amount of curvature induced on the brane. It has been argued that this model suffers from a strong coupling problem [19, 20], although the jury is still out in some respect [21]. For their part, multigravity models are often plagued by ghosts [9]. One can construct a DGP-like model that exhibits *bi*gravity and is free from ghosts [22], but the strong coupling problem still looms large. The ghost condensate, on the other hand, is an exotic form of matter whose dispersion relation has the form $\omega^2 \propto k^4$, owing to Lorentz symmetry breaking. This model is very interesting, and can successfully describe cosmic acceleration. However, it is a low energy effective theory, and as yet we have no insight into the UV completion beyond the Lorentz symmetry breaking scale.

In this paper, we will suggest an alternative to each of the above. We will consider a single 3-brane that acts as a domain wall between two different five dimensional spacetimes. These bulk spacetimes will, in general, be solutions to Gauss-Bonnet gravity. This is the combination of the Einstein-Hilbert action and the Gauss-Bonnet term,

$$S = M^3 \int_{\mathcal{M}} \mathrm{d}^5x\sqrt{-g}\{R - 2\Lambda + \alpha\mathcal{L}_{\mathrm{GB}}\}, \tag{1}$$

where

$$\mathcal{L}_{\mathrm{GB}} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}. \tag{2}$$

In 4 dimensions, a linear combination of the Einstein tensor and the metric is the most general combination of tensors satisfying the following conditions

- it is symmetric.
- it depends only on the metric and its first two derivatives.
- it has vanishing divergence.

- it is linear in the second derivatives of the metric[1].

If we go to 5 or 6 dimensions, it turns out that these conditions are satisfied by a linear combination of the metric, the Einstein tensor, and the *Lovelock tensor* [23, 24]. The Lovelock tensor arises from the variation of the Gauss-Bonnet term in the above action (1). In this sense, Gauss-Bonnet gravity is the natural generalisation of Einstein gravity to higher dimensions.

String theory provides us with an even more compelling reason to study Gauss-Bonnet gravity, especially in a braneworld context. In the Regge slope (α') expansion of the heterotic string action, curvature squared terms appear as the leading order correction to Einstein gravity [25, 26]. Furthermore, for this theory of gravity to be ghost-free, the curvature squared terms must appear in the Gauss-Bonnet combination [27, 28, 29].

Brane cosmologies with and without a Gauss-Bonnet correction have been extensively studied (see, for example [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]). Generically, to obtain a de Sitter phase of cosmological expansion one needs to introduce a positive vacuum energy in the bulk or on the brane [37, 44, 45, 46]. The alternative is to include some induced brane curvature [13, 14, 15, 40]. It is our desire to avoid doing either of these for the reasons outlined above. We should note, however, that in [40], the authors also manage to avoid an initial singularity through the combined effect of the induced curvature and the Gauss-Bonnet bulk. Although this will not happen in any of our models, we will suggest an alternative means of doing this in section 5. In [43], cosmic acceleration is achieved by considering negative Gauss-Bonnet coupling ($\alpha < 0$), but this is not well motivated by string theory, and has problems with stability [47].

The key feature in our model is the asymmetry across the brane. This means that the parameters in our theory can differ on either side of the brane. This has previously been applied to the bulk energy momemtum and the bulk Weyl tensor [33, 38, 39, 48, 49, 50, 51, 52, 53, 54, 55]. Here we will also apply it to the gravitational couplings, as in [56]. There are at least two ways in which this asymmetry might arise. Firstly, suppose we have some sort of wine bottle shaped compactification down to 5 dimensions. In the effective theory, the Planck scale at the fat end of the bottle will be less than that at the thin end. Secondly, in [57, 58] we showed how to construct a domain wall living entirely on the brane. In some cases [58], the Planck scale on the brane differed on either side of the domain wall.

The jump in the gravitational couplings leads to weighted junction conditions across the brane. In the absence of any vacuum energy on the brane, it is this fact that enables us to find late time de Sitter solutions, with $H^2 \sim (constant)^2$. However, this alone is not enough to describe a realistic cosmology. The accelerated expansion must be preceded by an era of matter and radiation domination, with $H^2 \propto \rho$, all the way back to nucleosynthesis. We will show that this can also be achieved, at least in certain parameter regions. This is perhaps the most remarkable aspect of our work.

The rest of this paper is organised as follows: in section 2 we will introduce our asymmetric brane action, and show how to construct a homogeneous and isotropic

[1] In 4 dimensions this condition is actually implied by the other three.

braneworld. In section 3, we will study a generalized Randall-Sundrum model, for which the bare cosmological constants are negative ($\Lambda < 0$) and there are no Gauss-Bonnet couplings ($\alpha = 0$). In section 4 we will consider a more stringy example. We will assume that there are no bare cosmological constants ($\Lambda = 0$), so that the bulk action contains only the Ricci scalar and Gauss-Bonnet terms. This is what we would expect from the slope expansion of heterotic string theory. Section 5 contains some concluding remarks.

2 Equations of motion

Consider two 5 dimensional spacetimes, $\mathcal{M}_1$ and $\mathcal{M}_2$, separated by a domain wall. The domain wall is a 3-brane corresponding to our universe. In general, $\mathcal{M}_i$ is a solution to Gauss-Bonnet gravity with a (bare) cosmological constant, Λ_i, and Gauss-Bonnet coupling α_i. The 5-dimensional Planck scale in this region will be given by M_i. We will not assume that there is $\mathbb{Z}_2$ symmetry across the brane, so that the fundamental parameters of our theory can differ on either side of the brane. This scenario is described by the following action,

$$S = S_{\text{grav}} + S_{\text{brane}}, \tag{3}$$

where

$$\begin{aligned} S_{\text{grav}} &= \sum_{i=1,2} M_i^3 \int_{\mathcal{M}_i} \mathrm{d}^5x\sqrt{-g}\{R - 2\Lambda_i + \alpha_i \mathcal{L}_{\text{GB}}\} + \int_{\partial\mathcal{M}_i} \text{boundary terms} && (4)\\ S_{\text{brane}} &= \int_{\text{brane}} d^4x\sqrt{-h}\,\mathcal{L}_{\text{brane}}. && (5) \end{aligned}$$

The boundary integrals in S_{grav} are required for a well defined action principle [59] (see also [60]). We denote the bulk metric and the brane metric by g_{ab} and h_{ab} respectively. $\mathcal{L}_{\text{brane}}$ describes the matter content on the brane. We will assume that this is made up of ordinary matter and radiation, so that there is no vacuum energy. In other words, there is no brane tension.

2.1 The bulk

In $\mathcal{M}_i$, the bulk equations of motion are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda_i g_{ab} + \alpha_i\left\{\frac{1}{2}\mathcal{L}_{\text{GB}}\, g_{ab} - 2RR_{ab} + 4R_{ac}R_b{}^c + 4R_{acbd}R^{cd} - 2R_{acde}R_b{}^{cde}\right\} \tag{6}$$

For the time being, let us drop the index i, as the following analysis will apply on both sides of the brane. We will put it back in when necessary.

If we demand that the bulk contains 3-dimensional spatial sections of constant curvature, we find the following solutions [38, 47, 61],

$$\mathrm{d}s^2 = -h(a)\mathrm{d}t^2 + \frac{\mathrm{d}a^2}{h(a)} + a^2\mathrm{d}\mathbf{x}_\kappa^2, \tag{7}$$

where

$$h(a) = \kappa + \frac{a^2}{4\alpha}\left(1 \pm \xi(a)\right) \quad \text{with} \quad \xi(a) = \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha\mu}{a^4}}. \tag{8}$$

For $\kappa = 1, 0, -1$, $\mathrm{d}\mathbf{x}_\kappa^2$ is the metric on a unit 3-sphere, plane, and hyperboloid respectively. μ is a constant of integration. Other solutions do exist for special values of κ, Λ, and α [38], but we will not consider them here.

For the metric to be real for all $0 \leqslant a < \infty$, we clearly require that

$$1 + \frac{4\alpha\Lambda}{3} \geqslant 0 \tag{9}$$

$$\alpha\mu \geqslant 0 \tag{10}$$

Furthermore, the mass of the spacetime is given by [62, 63, 64, 65]

$$m = 3M^3 V\mu, \tag{11}$$

where V is the volume of the homogeneous sections. In order to avoid a classical instability, we must have $\mu \geqslant 0$. This is in some sense counter-intuitive as the (+) branch of (8) then asymptotes to a Schwarzschild metric with *negative* mass, if one uses the standard ADM formula for Einstein gravity [66, 67].

From equation (10), we see that for $\mu > 0$, we must have $\alpha \geqslant 0$, which is consistent with string theory. On a less positive note, we also find that the metric has a singularity at $a = 0$. For the (-) branch, this singularity is covered by an event horizon. This is not the case for the (+) branch. To shield this naked singularity we must cut the spacetime off at some small value of a. This can be done by introducing a second brane at, say, $a \sim M^{-1}$ [68].

2.2 The brane

In order to construct a brane, we glue a solution in $\mathcal{M}_1$ to a solution in $\mathcal{M}_2$, with the brane forming the common boundary. Let us describe this in more detail. In $\mathcal{M}_i$, the boundary, $\partial\mathcal{M}_i$, is given by the section $(t_i(\tau), a_i(\tau), \mathbf{x}^\mu)$ of the bulk metric. The parameter τ is the proper time of an observer comoving with the boundary, so that

$$-h_i(a_i)\dot{t}^2 + \frac{\dot{a}^2}{h_i(a_i)} = -1, \tag{12}$$

where overdot corresponds to differentiation with respect to τ. The *outward* pointing unit normal to $\partial\mathcal{M}_i$ is now given by

$$n_a = \theta_i(-\dot{a}_i(\tau), \dot{t}_i(\tau), \mathbf{0}) \tag{13}$$

where $\theta_i = \pm 1$. For $\theta_i = -1$, $\mathcal{M}_i$ corresponds to $a_i(\tau) < a < \infty$, whereas for $\theta_i = 1$, $\mathcal{M}_i$ corresponds to $0 \leqslant a < a_i(\tau)$.

The induced metric on $\partial\mathcal{M}_i$ is that of a FRW universe,

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a_i(\tau)^2 \mathrm{d}\mathbf{x}_\kappa^2, \tag{14}$$

Since the brane coincides with both boundaries, the metric on the brane is only well defined when $a_1(\tau) = a_2(\tau) = a(\tau)$. The Hubble parameter on the brane is now defined by $H = \dot{a}/a$.

The dynamics of the brane are determined by the junction conditions for a braneworld in Gauss-Bonnet gravity [60, 69]. This comes from varying the action (3) with respect to the brane metric. Given a quantity Z_i defined in $\mathcal{M}_i$, we shall henceforth write $\langle Z \rangle = (Z_1 + Z_2)/2$, for the average across the brane, and $\Delta Z = Z_1 - Z_2$, for the difference. The brane equations of motion are given by [60, 69]

$$2\langle X_{ab}\rangle = T_{ab} - \frac{1}{3}T h_{ab} \tag{15}$$

where, suppressing the index i,

$$X_{ab} = 2M^3\left[K_{ab} + 2\alpha\left(Q_{ab} - \frac{2}{9}Q h_{ab}\right)\right]. \tag{16}$$

Here,

$$\begin{aligned} Q_{ab} \;&=\; 2KK_{ac}K_b{}^c - 2K_{ac}K^{cd}K_{db} + K_{ab}(K_{cd}K^{cd} - K^2) \\ &\quad +2K\mathcal{R}_{ab} + \mathcal{R}K_{ab} - 2K^{cd}\mathcal{R}_{cadb} - 4\mathcal{R}_{ac}K_b{}^c \end{aligned} \tag{17}$$

and $K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)}$ is the extrinsic curvature of the brane in $\mathcal{M}$. $\mathcal{R}_{abcd}$ is the Riemann tensor on the brane, constructed from the induced metric h_{ab}.

The energy-momentum tensor on the brane is given by

$$T_{ab} = -\frac{2}{\sqrt{-h}}\frac{\delta S_{\text{brane}}}{\delta h^{ab}}. \tag{18}$$

Since the brane is homogeneous and isotropic

$$T_{ab} = (\rho + p)\tau_a\tau_b + p h_{ab}, \tag{19}$$

where ρ is the energy density, p is the pressure, and τ^a is the velocity of a comoving observer. Note that in $\mathcal{M}_i$, $\tau^a = (\dot{t}_i(\tau), \dot{a}(\tau), \mathbf{0})$, and recall that the unit normal to $\partial\mathcal{M}_i$ is $n_a = \theta_i(-\dot{a}(\tau), \dot{t}_i(\tau), \mathbf{0})$. We now evaluate the spatial components of (15) to give

$$2\left\langle \theta M^3 \frac{h\dot{t}}{a}\left[1 - \frac{4}{3}\alpha\left(\frac{h\dot{t}}{a}\right)^2 + 4\alpha\left(H^2 + \frac{\kappa}{a^2}\right)\right]\right\rangle = \frac{\rho}{6}. \tag{20}$$

Making use of equation (12), we can simplify this expression to give

$$2\langle \theta F(H^2)\rangle = \frac{\rho}{6} \tag{21}$$

where

$$F(H^2) = M^3\sqrt{\frac{h}{a^2} + H^2}\left[1 + \frac{8}{3}\alpha H^2 + \frac{4}{3}\alpha\left(\frac{3\kappa - h}{a^2}\right)\right] \tag{22}$$

Equation (21) suggests that the brane dynamics will depend crucially on the relative signs of θ_1 and θ_2. We shall therefore consider each case separately. For $\theta_1 = \theta_2 = \theta$, we have

$$\theta\frac{\rho}{6} = G_+(H^2) \equiv 2\langle F(H^2)\rangle \tag{23}$$

whereas for $\theta_1 = -\theta_2 = \theta$, we have

$$\theta\frac{\rho}{6} = G_-(H^2) \equiv \Delta F(H^2) \tag{24}$$

These equations are very complicated. However, we can, in principle, analyse their behaviour, particularly at late times. Recall that we are assuming that there is *no* vacuum energy on the brane. We might naively expect this to prohibit cosmic acceleration at late times. In the examples that follow, we will show that this expectation is wrong, and that we *can* get $H^2 \sim (constant)^2$, for large a. Furthermore, we will show that, in certain parameter regions, this is preceded by an era of matter and radiation domination, with $H^2 \propto \rho$ as far back as nucleosynthesis.

The examples we will consider have been chosen both for interest and simplicity. These are the generalized Randall-Sundrum model and the stringy model described in the introduction.

3 The generalized Randall-Sundrum model

In this section we will consider the generalized RS model, for which

$$\alpha_i = 0 \qquad \Lambda_i = -\frac{6}{l_i^2} \tag{25}$$

If we assume that $M_1 \neq M_2$, we can, without loss of generality, take $M_1 > M_2 > 0$. For the "symmetric" scenario ($\theta_1 = \theta_2$), this corresponds to the model discussed in [56].

Only the (-) branch of (8) is well-defined. It is reduced to the AdS-Schwarzschild metric

$$h(a) = \kappa + \frac{a^2}{l^2} - \frac{\mu}{a^2} \tag{26}$$

The equations of motion are given by (23) and (24) with

$$F(H^2) = M^3\sqrt{\frac{1}{l^2} - \frac{\mu}{a^4} + \frac{\kappa}{a^2} + H^2} \tag{27}$$

We shall begin by looking for late time de-Sitter solutions, in order to describe the current cosmic acceleration. We will assume that $a \to \infty$ at late times. In this limit,

$$F(H^2) \to M^3\sqrt{\frac{1}{l^2} + H^2}, \qquad \rho \to 0 \tag{28}$$

where we have used the fact that there is no vacuum energy contained in ρ. Let us start with the "symmetric" equation of motion (23). At late times, it reads

$$0 = G_+(H^2) \equiv 2\langle F(H^2)\rangle \tag{29}$$

It is easy to see that

$$H^2 > 0 \quad \Longrightarrow \quad G_+(H^2) > 2\langle M^3/l\rangle > 0. \tag{30}$$

This means that (29) has no solutions in $H^2 > 0$. We conclude that it is impossible to get late time de Sitter expansion when $\theta_1 = \theta_2$.

Now consider the "antisymmetric" equation of motion

$$0 = G_-(H^2) \equiv \Delta F(H^2). \tag{31}$$

This is easily solved to give

$$H^2 \sim H_0^2 = -\frac{\Delta(M^6/l^2)}{\Delta M^6} \tag{32}$$

Therefore, when $\theta_1 = -\theta_2$, we have late time de Sitter expansion provided $\Delta(M^6/l^2) < 0$.

We are ready to ask whether or not this de Sitter phase is preceded by a period of matter/radiation domination, with $H^2 \propto \rho$. Consider the "antisymmetric" equation of motion (24) at smaller values of a. We can manipulate this equation to give a quadratic in $H^2 + \frac{\kappa}{a^2}$,

$$\left(\Delta M^6\right)^2 \left(H^2 + \frac{\kappa}{a^2}\right)^2 + \left[2\Delta\left[M^6 V(a)\right]\Delta M^6 - \frac{\rho^2}{9}\langle M^6\rangle\right]\left(H^2 + \frac{\kappa}{a^2}\right)$$
$$\left[\Delta\left(M^6 V(a)\right)\right]^2 + \frac{\rho^2}{36}\left[\frac{\rho^2}{36} - 4\Big\langle M^6 V(a)\Big\rangle\right] = 0 \tag{33}$$

where

$$V(a) = \frac{1}{l^2} - \frac{\mu}{a^4} \tag{34}$$

Now solve this quadratic to derive the Friedmann equation

$$H^2 + \frac{\kappa}{a^2} = -\frac{\Delta\left[M^6 V(a)\right]}{\Delta M^6} + \rho^2 \frac{\langle M^6\rangle}{18\left(\Delta M^6\right)^2} \pm \frac{M_1^3 M_2^3 \rho}{3\left(\Delta M^6\right)^2}\sqrt{\frac{\rho^2}{36} - \Delta M^6 \Delta V} \tag{35}$$

The choice of root corresponds to a choice of $\theta = \pm 1$. When

$$|\Delta\mu/a^4| \ll |\Delta(1/l^2)| \qquad \text{and} \qquad \rho \ll \rho_{\max} = \frac{6M_1^3 M_2^3}{\langle M^6\rangle}\sqrt{-\Delta(M^6)\Delta(1/l^2)} \tag{36}$$

The Friedmann equation approximates to

$$H^2 + \frac{\kappa}{a^2} \approx H_0^2 + \frac{\Delta(M^6\mu)}{\Delta M^6}\frac{1}{a^4} \pm \frac{M_1^3 M_2^3\sqrt{-\Delta(M^6)\Delta(1/l^2)}}{3\left(\Delta M^6\right)^2}\rho \tag{37}$$

Recall that for $H_0^2 > 0$, we chose $\Delta M > 0$, and demanded that $\Delta\left(M^6/l^2\right) < 0$. This ensures that the square root in equation (37) is real. In order to reproduce the

Friedmann equation of the standard cosmology, we must take the positive square root in (37), which corresponds to $\theta = 1$. We then find that

$$H^2 + \frac{\kappa}{a^2} \approx H_0^2 + \frac{\Delta(M^6\mu)}{\Delta M^6}\frac{1}{a^4} + \frac{\rho}{6M_{\rm b}^2} \tag{38}$$

where the four-dimensional Planck mass on the brane is given by

$$M_{\rm b}^2 = \frac{\left(\Delta M^6\right)^2}{2M_1^3M_2^3\sqrt{-\Delta(M^6)\Delta(1/l^2)}} \tag{39}$$

Note that

$$\rho_{\rm max} = \frac{3\left(\Delta M^6\right)^2}{\langle M^6\rangle M_{\rm b}^2} \tag{40}$$

The μ/a^4 term in (37) comes from the bulk, and behaves like a form of dark radiation. We can interpret it holographically as the energy density of a conformal field theory dual to the bulk [37, 46, 70, 71]. For simplicity let us assume that this contribution is always small compared to the energy density on the brane, so that the Friedmann equation behaves as

$$H^2 + \frac{\kappa}{a^2} \approx H_0^2 + \frac{\rho}{6M_{\rm b}^2} \tag{41}$$

When does this equation describe real physics? To predict the current cosmic acceleration, we need $H_0^2 \sim 10^{-68}$ (eV)^2. Prior to this, we need $H^2 \sim \rho/6m_{pl}^2$, where $m_{pl} \sim 10^{19}$ GeV. This must be the case as far back as nucleosynthesis, at which point $\rho = \rho_{\rm NS} \sim 10^{24}$ (eV)^4. For equation (41) to be physical, we therefore require that $M_{\rm b} \sim m_{pl}$, and

$$\frac{\rho_\Lambda}{\rho_{\rm max}} \ll \frac{\rho_\Lambda}{\rho_{\rm NS}} \sim 10^{-36} \tag{42}$$

where $\rho_\Lambda = 6M_{\rm b}^2H_0^2 \sim 10^{-12}$ (eV)^4. Note that the scale of curvature in the spatial sections is observed to be very close to zero. This means that we can ignore the κ/a^2 contribution in equation (41).

Suppose we take

$$M_1 = (1+\lambda)^{1/6}M_2, \qquad l_1 = (1+\lambda+\epsilon)^{1/2}l_2 \tag{43}$$

where $\lambda > 0$ is of order one, and $0 < \epsilon \lesssim 10^{-36}$. We find that

$$M_{\rm b}^2 \approx \frac{\lambda}{2}M_2^3l_2, \qquad H_0^2 \approx \frac{\epsilon}{\lambda(1+\lambda)}\frac{1}{l_2^2} \tag{44}$$

and the condition (42) holds. As an example, consider $\epsilon \sim 10^{-36}$. If we take $1/l_2 \sim 10^{-16}$ eV and $M_2 \sim 10$ TeV, we obtain precisely the desired cosmology from nucelosynthesis onwards.

4 The stringy model

Our next example is motivated by the slope expansion in heterotic string theory. There is no bare cosmological constant in this expansion and the slope parameter (α') is positive. We therefore take

$$\Lambda_i = 0, \qquad \alpha_i > 0 \tag{45}$$

As before, we would expect μ_i to enter the dynamics as some form of dark radiation (see, for example [42]). For the (+) branch with $\mu_i > 0$, we must introduce a second brane to shield the singularity. If this is done at small enough a, we would not expect it to significantly affect the dynamics of the main cosmological brane. In any case, let us avoid such complications by setting $\mu_i = 0$. We will do this even for the (-) branch to keep our analysis tidy.

For the (+) branch, we find that the bulk metric is given by

$$h(a) = \kappa + \frac{a^2}{2\alpha} \tag{46}$$

For the (-) branch, we are only allowed $\kappa = 1$, so that

$$h(a) = 1 \tag{47}$$

Note that the (+) branch is not well defined at $\alpha = 0$. For this reason, it represents a significant departure from Einstein gravity, and is of particular interest. We will focus on this solution presently.

4.1 The (+) branch

For the (+) branch, note that the metric corresponds to anti-de Sitter space with the appropriate slicing (depending on κ). The effective cosmological constant is given by

$$\Lambda_{\text{eff}} = -\frac{3}{\alpha}. \tag{48}$$

This may be surprising given that we set the bare cosmological constant to zero. The brane equations of motion are, of course, given by (23) and (24), but with

$$F(H^2) = \frac{1}{3}M^3\sqrt{\frac{1}{2\alpha} + H^2 + \frac{\kappa}{a^2}}\left[1 + 8\alpha\left(H^2 + \frac{\kappa}{a^2}\right)\right] \tag{49}$$

Again, we begin by looking for late time de Sitter solutions. As $a \to \infty$,

$$F(H^2) \to \frac{1}{3}M^3\sqrt{\frac{1}{2\alpha} + H^2}\left[1 + 8\alpha H^2\right], \qquad \rho \to 0 \tag{50}$$

The "symmetric" equation of motion (23) has no solution, since

$$H^2 > 0, \quad \alpha > 0 \implies G_+(H^2) > \sqrt{2}\langle M^3/\sqrt{\alpha}\rangle > 0 \tag{51}$$

As before, we conclude that late time de Sitter expansion is impossible when $\theta_1 = \theta_2$.

Now consider the "antisymmetric" equation of motion (24). Since $G_+(H^2)$ is never zero in $H^2 > 0$, the equation

$$0 = P(H^2) \equiv G_+(H^2)G_-(H^2) = \Delta\left[F(H^2)\right]^2 \tag{52}$$

must have the same roots ($H_0^2 > 0$) as $G_-(H^2) = 0$. $P(H^2)$ is a cubic in H^2,

$$P(H^2) = \frac{64}{9}\Delta(M^6\alpha^2)H^6 + \frac{16}{3}\Delta(M^6\alpha)H^4 + \Delta(M^6)H^2 + \frac{1}{18}\Delta\left(\frac{M^6}{\alpha}\right) \tag{53}$$

To ensure the existence of a real solution, $P(H_0^2) = 0$, we demand that

$$\Delta\left(\frac{M^6}{\alpha}\right) < 0, \qquad \Delta(M^6\alpha) > 0 \tag{54}$$

This is a sufficient (although perhaps not a necessary) condition. To see this note that we must have $\alpha_1 > \alpha_2$ for the two inequalities in (54) to be consistent. In addition, we deduce that $\Delta(M^6\alpha^2) > 0$. We now see that $P(0) < 0$, whereas $P(H^2) \to +\infty$ as $H^2 \to +\infty$. By the intermediate value theorem, there exists $0 < H_0^2 < \infty$, such that $P(H_0^2) = 0$. For $\theta_1 = -\theta_2$, we conclude that we have late time de Sitter expansion whenever the condition (54) holds.

We now examine the behaviour of the "antisymmetric" equation of motion (24) at smaller values of a. This will provide us with an estimate for H_0^2, and enable us to check for a period of matter/radiation domination with $H^2 \propto \rho$.

We can manipulate equation (24) to obtain a polynomial in $H^2 + \frac{\kappa}{a^2}$ of degree six. It is therefore very difficult to find solutions so we adopt another approach. We will expand $G_-(H^2)$ as a Taylor series, about the point $H^2 + \frac{\kappa}{a^2} = 0$,

$$\theta\frac{\rho}{6} = \frac{1}{3\sqrt{2}}\Delta\left(\frac{M^3}{\sqrt{\alpha}}\right) + \frac{3}{\sqrt{2}}\Delta\left(M^3\sqrt{\alpha}\right)\left(H^2 + \frac{\kappa}{a^2}\right) + \text{higher order terms} \tag{55}$$

This expansion is valid when

$$H^2 + \frac{\kappa}{a^2} \ll \min\left\{\frac{1}{\alpha_1},\ \frac{1}{\alpha_2},\ \left|\frac{\Delta\left(M^3\sqrt{\alpha}\right)}{\Delta\left(M^3\alpha\sqrt{\alpha}\right)}\right|\right\} \tag{56}$$

Let us assume, for the time being, that this holds. We will come back to it later. Ignoring the higher order terms, we rearrange equation (55) to give

$$H^2 + \frac{\kappa}{a^2} \approx H_0^2 + \frac{\rho}{6M_{\rm b}^2} \tag{57}$$

where

$$H_0^2 = -\frac{\Delta\left(M^3/\sqrt{\alpha}\right)}{9\Delta\left(M^3\sqrt{\alpha}\right)}, \qquad M_{\rm b}^2 = \frac{3\Delta\left(M^3\sqrt{\alpha}\right)}{\sqrt{2}} \tag{58}$$

If we impose condition (54), we have $H_0^2 > 0$. Furthermore, to ensure $M_\mathrm{b}^2 > 0$, we have chosen $\theta = 1$. Since we must now have $\alpha_1 > \alpha_2$, it is fairly easy to show that

$$\left|\frac{\Delta\left(M^3\sqrt{\alpha}\right)}{\Delta\left(M^3\alpha\sqrt{\alpha}\right)}\right| < \frac{1}{\alpha_1} < \frac{1}{\alpha_2} \tag{59}$$

Our validity condition now reads

$$H^2 + \frac{\kappa}{a^2} \ll \left|\frac{\Delta\left(M^3\sqrt{\alpha}\right)}{\Delta\left(M^3\alpha\sqrt{\alpha}\right)}\right| \tag{60}$$

For $\rho \gg \rho_\Lambda = 6M_\mathrm{b}^2 H_0^2$, this translates into an upper bound on the energy density.

$$\rho \ll \rho_\mathrm{max} = 6M_\mathrm{b}^2\left|\frac{\Delta\left(M^3\sqrt{\alpha}\right)}{\Delta\left(M^3\alpha\sqrt{\alpha}\right)}\right| \tag{61}$$

For consistency with observations, we again require that $H_0^2 \sim 10^{-68}\ (\mathrm{eV})^2$, $M_\mathrm{b} \sim m_{pl}$, and

$$\frac{\rho_\Lambda}{\rho_\mathrm{max}} \ll \frac{\rho_\Lambda}{\rho_\mathrm{NS}} \sim 10^{-36} \tag{62}$$

where we remind the reader that $\rho_\mathrm{NS} \sim 10^{24}\ (\mathrm{eV})^4$ is the energy density at the time of nucleosynthesis.

Suppose we take

$$M_1 = (1+\lambda)^{1/6} M_2, \qquad \alpha_1 = \frac{1+\lambda}{(1-\epsilon)^2}\alpha_2 \tag{63}$$

where, again, $\lambda > 0$ is of order one, and $0 < \epsilon \lesssim 10^{-36}$. We find that

$$M_\mathrm{b}^2 \approx \frac{3\lambda}{\sqrt{2}} M_2^3\sqrt{\alpha_2}, \qquad H_0^2 \approx \frac{\epsilon}{9\lambda}\frac{1}{\alpha_2} \tag{64}$$

and the condition (62) holds. As an example, consider $M_2 \sim m_{pl}$, $\alpha_2 \sim 1/m_{pl}^2$, as we might expect from string theory. If we take $\epsilon \sim 10^{-124}$, we obtain the desired cosmology after nucleosynthesis, In fact, with this level of fine tuning, we will still have good agreement with the standard cosmology at much earlier times.

We end this section with a few comments about quantum stability, and the absence of a zero mode. In [39], there are examples of flat branes, where the bulk metric corresponds to a $(+)$ branch. Transverse-tracefree perturbations about these solutions include a normalisable zero mode. This mode turns out to be a ghost in the effective theory. We might be worried that this will also happen here, in the limit that $H_0^2 \to 0$. However, in our model, note that $\theta_1 = -\theta_2 = 1$. This means that $\mathcal{M}_1$ corresponds to $0 \leqslant a < a(\tau)$, and $\mathcal{M}_2$ corresponds to $a(\tau) < a < \infty$. The volume of our background is therefore infinite, so there is no normalisable zero mode. We cannot, therefore, apply the results of [39].

The absence of a zero mode has another important implication: gravity cannot be localised. However, given the close relationship between models with infra-red modifications of gravity and cosmic acceleration (see, for example, [13]) we might

expect our model to exhibit *quasi*-localisation. This occurs in the DGP model, where a resonance of massive modes leads to four-dimensional brane gravity at intermediate scales. Will something similar happen here? Quite possibly. Roughly speaking, in the antisymmetric case, we have finite volume on one side of the brane, and infinite volume on the other. On the finite side we have a localised graviton, whereas on the infinite side we have a non-localised graviton. At intermediate scales it could be that the finite side dominates, so that gravity appears localised on the brane. This would be consistent with the fact that we reproduce the standard cosmology as far back as nucleosynthesis. Of course, a more thorough investigation is clearly required.

4.2 The (-) branch

In this section, we consider the (-) branch, for which $\kappa = 1,\ h(a) = 1$. This corresponds to flat space in the bulk. The equations of motion are given by (23) and (24) with

$$F(H^2) = M^3\sqrt{\frac{1}{a^2}+H^2}\left[1+\frac{8}{3}\alpha\left(\frac{1}{a^2}+H^2\right)\right] \tag{65}$$

Now look for late de Sitter behaviour. As $a \to \infty$

$$F(H^2) \to M^3 H\left[1+\frac{8}{3}\alpha H^2\right], \qquad \rho \to 0 \tag{66}$$

It is easy to see that $G_+(H^2) = 0$ has no solution in $H^2 > 0$. This rules out the possibility of late time de Sitter when $\theta_1 = \theta_2$. Now consider $G_-(H^2) = 0$. This has the solution

$$H^2 = H_0^2 = -\frac{3\Delta M^3}{8\Delta(M^3\alpha)} \tag{67}$$

This can be made positive if we take

$$\Delta M^3 > 0, \qquad \Delta(M^3\alpha) < 0 \tag{68}$$

At smaller values of a, equation (24) is already a cubic in $\mathcal{H} = \sqrt{\frac{1}{a^2}+H^2}$,

$$\frac{8}{3}\Delta(M^3\alpha)\mathcal{H}^3 + \Delta M^3\mathcal{H} - \theta\frac{\rho}{6} = 0 \tag{69}$$

For $\theta = 1$, this equation has no roots with $\mathcal{H} \geqslant H_0$. We therefore choose $\theta = -1$, and find that the only real root with $\mathcal{H} \geqslant H_0$ is given by

$$\mathcal{H} = H_+ + H_- \tag{70}$$

where

$$H_\pm^3 = H_\rho^3 \pm \sqrt{H_\rho^6 - \frac{H_0^6}{27}}, \qquad -\frac{\pi}{3} \leqslant \arg(H_\pm) \leqslant \frac{\pi}{3} \tag{71}$$

and

$$H_\rho^3 = -\frac{\rho}{32\Delta(M^3\alpha)}. \tag{72}$$

During a matter/radiation dominated era, we expect $H_\rho \gg H_0$, so that

$$\mathcal{H} \approx 2^{1/3} H_\rho \tag{73}$$

This implies that

$$H^2 + \frac{1}{a^2} \propto \rho^{2/3} \tag{74}$$

which disagrees with the standard cosmology. We conclude that this model does give cosmic acceleration, but does not predict the correct physics beforehand. Perhaps this is not surprising given that we have a flat bulk. For gravity to be localised on the brane we would expect there to be a warp factor in the bulk metric.

5 Discussion

In this paper, we have considered the cosmology of an asymmetric 3-brane, sandwiched between two different bulk spacetimes. In general, the bulk spacetimes are solutions to Gauss-Bonnet gravity, with the gravitational couplings being allowed to differ on either side of the brane. This leads to weighted junction conditions across the brane, so that we naturally have many more solutions available.

We focussed on two special cases: a generalized Randall-Sundrum model and a stringy model. In the former, we switched off all higher derivative couplings, and set the bare cosmological constant to be negative. In the latter, we set the bare cosmological constant to be zero, and demanded that the Gauss-Bonnet couplings be positive, in accordance with string theory.

The generic behaviour of both models was the same, depending crucially on whether we were considering a "symmetric" or an "antisymmetric" scenario. For the symmetric scenario, a was a minimum (or a maximum) at the brane for both bulk spacetimes. For the antisymmetric scenario, a was a maximum at the brane for one side of the bulk, and a minimum for the other. The symmetric scenario did not permit de Sitter cosmologies, unlike the antisymmetric scenario. Naturally, to ensure that the late time acceleration agreed with observations, a degree of fine-tuning was required. What is interesting is that cosmic acceleration of any sort could be achieved without resorting to vacuum energy or induced curvature. It was made possible by the weighted junction conditions. Furthermore, for the antisymmetric case, the brane cosmology could be made to follow the standard cosmology all the way back to nucleosynthesis. This is our main result.

Finally, as promised, we will comment on the initial singularity. As they are, our models will predict a time of infinite energy density, so that our theory will eventually break down. We might hope that by adjusting the bulk mass parameters, we could enforce a "bounce" cosmology. This means that the scale factor has a non-zero minimum. However, the bulk mass looks like dark radiation on the brane, so it is difficult to see what new effect it could have. However, in Einstein gravity, bounce cosmologies *do* occur for branes moving in between *charged* black holes [71, 37]. We could consider the analagous situation in Gauss-Bonnet gravity coupled to

electromagnetism. Can the initial singularity be avoided without spoiling the late time behaviour? We will leave this for future research.

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